A Chaotic Expansion Formula for Approximating First-Order, Non-Linear Dynamical Systems

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ABSTRACT

A general chaotic expansion formula with variable coefficients is developed to approximate non-linear, dynamical systems. The initial conditions as well as other key inflection points of the sequence are used to linearly determine the coefficients of the expansion. An example of approximating a simple model of population dynamics is shown where the approximation shows an MSE of 0.14 over 5000 samples of the sequence.

1. INTRODUCTION

Chaotic data sequences are found to exist in nature and general mathematical formulations. There is a necessity to develop a deterministic formula that approximates such sequences, which may also lead to the development of the characteristic equation that underlies the data [1,2]. In this paper a simple expansion formula is presented for approximating first-order non-linear sequences, where the coefficients of the expansion formula are matched to the sequence through linear determination. The initial conditions and other values of the sequence are used in the linear determination of the coefficients that best fit to the data.

2. THE EXPANSION FORMULA

The expansion formula proposed was been determined experimentally through the numerical analysis of a chaotic oscillator [3] to be:

$$x_n = \sum_{i=1}^{1} k_i x_{n-i}^{p} - \sum_{i=2}^{p} k_i x_{n-i}^{p} \qquad (1)$$

Where x_n is the currently computed sequence value, x_{n-i} is the history of sequence values for historical index i, and k_i is the coefficient for each history sample x_{n-i} raised to the power p.

The general form of (1) is then a finite expansion of power sequences where the power p of each term is the same number of terms in the expansion. The accuracy of the approximation is based on p, which determines both the number of terms in the expansion and the power of each term.

The requirements for approximating a sequence is to linearly determine the coefficients k_i based on the expansion formula of (1):

$$K_i = X_n X_h^{-1} \qquad (2)$$

Where K_i is the coefficient matrix and X_n is the current value matrix that consists of a selection of values in the sequence which should include the initial conditions of the sequence based on the sensitivity of the system to initial conditions. X_h is the history matrix which are history values in the sequence raised to the power p, where the history values are the terms x^p_{n-i} shown in (1). K_i is then found as the product of the current value matrix X_n and the history matrix X_h.

2. APPROXIMATING A FIRST-ORDER SEQUENCE

We consider an example of approximating a simple, first-order sequence of population dynamics [1]:

$$x_{n+1} = ax_n(1 - x_n) \qquad (3)$$

We will refer to (3) as the target function and we note that it approaches a chaotic attractor at a = 3.8 with a corresponding initial condition $x_0 = 0.7$. To approximate this sequence with these conditions we use the first five points of the sequence, $\{x_0...x_5\}$ which includes the initial condition, in the solution of (2). We also use values throughout the sequence for the solution of (2) that would indicate changes or bifurcation on a phase diagram so that these are reflected in the expansion formula. The mathematical solution of (2) for expansion constants k_i as it applies to the simple first-order equation of (3) with $x_0 = 0.7$ is as follows:

$$k_1 = -.818002 \quad k_2 = -1.431836$$
$$k_3 = -1.204792 \quad k_4 = -.7801357 \qquad (4)$$

By incorporating the k_i constants above into (1) and using the same initial condition $x_0 = 0.7$, we find the mean-squared error by taking the square of the difference of the approximation expansion of (1) from the target function of (3) over 5000 sequence points starting with the initial condition. The MSE is found to be 0.14, and the standard deviation of the error terms found by taking the absolute value of the difference between the approximation (1) and target function (3) is 0.486 over 5000 sequence points. From this standard deviation of the error terms it is found that 82.6% of the error terms are less within one standard deviation of the mean of the target function. By randomly changing constants in (4), the MSE is found to increase into single or double digits greater than 1 and with a confidence interval from the standard deviation of error terms that is less than that for a normal distribution.

3. CONCLUSIONS

An expansion formula has been presented that approximates first-order chaotic sequences with a linearly determined fit of the coefficients of the expansion with the sequence data. An example of approximating of a first-order chaotic system of population dynamics has been demonstrated with reasonable results.

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